Defi: / Th: Cauchy's general Principle of convergence for sequence.

The necessary and sufficient condition for a sequence $<x_n>$ to be convergent is that for a given $\epsilon > 0$, however small, $\exists$ a positive integer n_0 & p such that

$$|x_{n+p} - x_n| < \epsilon \ \forall\ n \geq n_0,\ p \in N$$
(or $p \geq 1$)

or

A sequence is convergent iff it is Cauchy's sequence.

Proof:- Let $<x_n>$ is a convergent sequence and converges to l.

$\Rightarrow$ For a given $\epsilon_1 > 0$ (however small), $\exists\ n_0 \in N$ s.t. $|x_n - l| < \epsilon_1 \ \forall\ n \geq n_0$

take $\epsilon_1 = \epsilon/2$

$\Rightarrow |x_n - l| < \epsilon/2 \ \forall\ n \geq n_0$

$\Rightarrow |x_{n+p} - l| < \epsilon/2 \ \forall$ ~~[illegible]~~ $n+p \geq n_0$

$\Rightarrow |x_{n+p} - l| < \epsilon/2$ or $\forall\ n \geq n_0,\ p \in N$

$$\Rightarrow |x_{n+p} - x_n| = |(x_{n+p} - l) + (l - x_n)|$$
$$\leq |x_{n+p} - l| + |x_n - l|$$
$$< \epsilon/2 + \epsilon/2 \qquad \forall\ n \geq n_0 \ \&\ p \in N$$

$\Rightarrow |x_{n+p} - x_n| < \epsilon \quad \forall\, n \geq n_0,\ p \in N$

$\Rightarrow \langle x_n \rangle$ is Cauchy sequence.

The condition is sufficient :-

Let $\langle x_n \rangle$ be a Cauchy sequence.
But we know that every Cauchy sequence is bounded
$\Rightarrow \langle x_n \rangle$ is bounded.
But we also know that every bounded sequence has a convergent subsequence
$\Rightarrow \langle x_n \rangle$ has a convergent subsequence
Let $\langle x_{n_k} \rangle$ is conv. subsequence of $\langle x_n \rangle$

$\Rightarrow \lim\limits_{k} x_{n_k} = l$

$\Rightarrow \lim\limits_{n} x_n = l$

$\Rightarrow \langle x_n \rangle$ is convergent sequence.

H.P.

Th Sandwich Theorem:

If $\langle x_n \rangle$, $\langle y_n \rangle$ and $\langle u_n \rangle$ are three sequences ~~are~~ having the property: For some positive integer

(i) $x_n \le u_n \le y_n \quad \forall\ n > k$

(ii) and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = l$

then $\lim_{n\to\infty} u_n = l$

Some important properties

(a) If for any sequence $\langle x_n \rangle$

$\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = \lambda \ (|\lambda|<1)$, then

$$\lim_{n\to\infty} x_n = 0$$

(b) If $\langle x_n \rangle$ be a sequence of positive terms and

$\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = l$, then

$$\lim_{n\to\infty} (x_n)^{\frac{1}{n}} = l$$

Ex-1 Let $x_n = \lfloor n \Rightarrow x_{n+1} = \lfloor n+1$

$\Rightarrow \frac{x_{n+1}}{x_n} = n+1$

$\Rightarrow \lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = \lim_{n\to\infty}(n+1) \Rightarrow \infty$

$\Rightarrow \lim_{n\to\infty}(x_n)^{1/n} \Rightarrow \infty \Rightarrow \lim_{n\to\infty}(\lfloor n)^{\frac{1}{n}} \Rightarrow \infty$

Ex.2 Similarly $\lim_{n\to\infty}\left(\frac{n^n}{\lfloor n}\right)^{\frac{1}{n}} = e$

Let $x_n = \frac{n^n}{\lfloor n}$

$\Rightarrow \frac{x_{n+1}}{x_n} = \left(1+\frac{1}{n}\right)^n$

$\Rightarrow \lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$= e$

$\Rightarrow \lim_{n\to\infty}\left(\frac{x^n}{\lfloor n}\right)^{\frac{1}{n}} = e$

H.P